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How singular functions define distributions

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Abstract
Following Dirac, Schwartz, and others, distributions are well understood (and widely used in physics) as 'generalized functions'. However, a function with a nonintegrable singularity does not define a distribution automatically or unambiguously. We review a variety of ways in which such distributions can be defined, sometimes with inequivalent results, or results containing arbitrary constants. We give special attention to the function $\operatorname{cosech}^2 x$ and its semiclassical scaling limit, which have recently attracted some attention in statistical mechanics.

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1. Introduction

It is by now well understood that Dirac's delta function is not really a function, but rather a distribution, or linear functional. That is, it is an operation on functions defined by the formula

$$\delta[\phi] \equiv \phi(0). \tag{1}$$

An ordinary continuous function f also defines a distribution, via

$$f[\phi] \equiv \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x. \tag{2}$$

But there is no function $\delta(x)$ such that

$$\delta[\phi] = \int_{-\infty}^{\infty} \delta(x)\phi(x)\,\mathrm{d}x. \tag{3}$$

Thus, today 'everyone knows' that not every distribution has a function corresponding to it—the delta distribution being the most familiar example. What is less widely appreciated is that not every function defines a distribution, at least not automatically. The problem is that if the function f is singular at certain points, then integral (2) may not converge. At best, further discussion is needed before (2) can be interpreted as a linear functional. This paper is devoted to an exposition of the state of the art in this area, with special attention to a particular example that has recently been discussed in the physics literature [8]. For several standard examples, we have stated results without showing all the calculational steps that lead to them; for the details please see [5] and [4] or other standard treatises. We have made every effort to make the exposition accessible to a wide audience, without sacrificing mathematical correctness.

2. The regularization problem

We start by making (2) more precise. Let U be an open region in $\mathbb{R}^n$, and let f be a function in $L^1_{\text{loc}}(U)$; that is, the (Lebesgue) integral

$$\int_S |f(\boldsymbol{x})|\,\mathrm{d}^n\boldsymbol{x} \tag{4}$$

is defined and finite for every closed, bounded subset S of U. (In particular, any continuous function will do, even if it fails to approach 0 at infinity or becomes unbounded at the boundary of U.) Then f *canonically and uniquely* defines a distribution $\tilde{f}$ in the distribution space $\mathcal{D}'(U)$ via the formula

$$\tilde{f}[\phi] \equiv \int_U f(\boldsymbol{x})\phi(\boldsymbol{x})\,\mathrm{d}^n x \qquad \text{for all } \phi \in \mathcal{D}(U). \tag{5}$$

Here $\mathcal{D}(U)$ is the space of all compactly supported smooth functions defined on U; that is, ϕ is differentiable arbitrarily many times and equals zero outside some closed, bounded set S inside U. By putting such restrictive conditions on the 'test functions' ϕ, one maximizes the class of functions f for which definitions such as (5) or (1) make sense. A distribution is also required to be continuous in a certain sense in its dependence on ϕ, but we need not dwell on that technicality here. The tilde distinguishing the distribution $\tilde{f}$ from the function f is routinely left off when there is no likelihood of confusion.

The alternative notation

$$\langle \tilde{f}, \phi\rangle \equiv \tilde{f}[\phi] \tag{6}$$

is convenient and widely used, so we will switch to it henceforth. With definition (5), $\langle f, \phi\rangle$ reduces to the ordinary inner product of two functions when f and ϕ are both *real*, square-integrable functions. However, it must be understood that there is *no* complex conjugation expressed in (5) or implied in (6).

Now we turn to the situation where f has a nonintegrable singularity, so that it fails to belong to $L^1_{\text{loc}}(U)$. For simplicity we consider a function $f(x)$ of one variable, which is singular at only one point, x_0. (The generalization to more than one singular point is immediate, and that to more than one variable is straightforward.) Thus $f \in L^1_{\text{loc}}(U\backslash\{x_0\})$. Four examples for f, all with $x_0 = 0$, are

$$\frac{1}{x}, \quad \frac{1}{x^2}, \quad \coth x \equiv \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} \quad \operatorname{cosech}^2 x \equiv \frac{4}{(\mathrm{e}^x - \mathrm{e}^{-x})^2}. \tag{7}$$

Definition (5) continues to make sense as long as ϕ is in $\mathcal{D}(U\backslash\{x_0\})$—that is, in effect, whenever ϕ vanishes in a neighbourhood of x_0. The crucial definition of the subject is the following:

> A distribution $\tilde{f} \in \mathcal{D}'(U)$ is a *regularization* of f if (5) holds whenever the integral converges.

Remark. A similar definition applies to any distribution $f \in \mathcal{D}'(U\backslash\{x_0\})$ in place of the function $f \in L^1_{\text{loc}}(U\backslash\{x_0\})$. Namely, $\tilde{f}$, a distribution defined on all $\phi \in \mathcal{D}(U)$, is a regularization of f if

$$\langle \tilde{f}, \phi\rangle = \langle f, \phi\rangle \qquad \text{for all } \phi \in \mathcal{D}(U\backslash\{x_0\}). \tag{8}$$

However, our definition above for functions is slightly stronger, since it requires that the restricted and extended distributions agree ‘whenever the integral converges’, which might happen even if ϕ does not belong to $\mathcal{D}(U\backslash\{x_0\})$.

The central fact that we wish to emphasize is that *even if* $\tilde{f}$ *exists*, *there is no unique*, *canonical way to construct it*. There is an inherent ambiguity in $\tilde{f}$, represented by a certain number of undetermined constants describing how $\tilde{f}$ acts on test functions that do *not* vanish in a neighbourhood of x_0. This phenomenon arises in *renormalization* in quantum field theory. Renormalized coupling constants are not determined by the theory, but must be fixed by experiment. In some approaches to renormalization [1, 11] the renormalized coupling constants arise *precisely* as undetermined constants in regularized distributions. (In fact, what mathematicians dealing with distributions call ‘regularization’ is more closely related to ‘renormalization’ than to ‘regularization’ as field theorists use the word, referring to cut-offs introduced at intermediate steps for technical convenience.)

Actually, it can be shown that there does not exist any *continuous* linear mapping ρ from $\mathcal{D}'(U\backslash\{x_0\})$ into $\mathcal{D}'(U)$ with the desired property that

$$\pi(\rho(f)) = f \tag{9}$$

for all $f \in \mathcal{D}'(U\backslash\{x_0\})$, where $\pi : \mathcal{D}'(U) \to \mathcal{D}'(U\backslash\{x_0\})$ is the natural restriction mapping (i.e. $\pi(\tilde{f})$ is just $\tilde{f}$ but applied only to functions that vanish in a neighbourhood of the singular point). Therefore, there is no *canonical* way to define a regularization for *all* singular functions.

Nevertheless, there are numerous prescriptions in the mathematical and physical literature for defining unique regularizations of particular singular functions or classes of singular functions in a ‘natural’ way. We shall discuss three of these in turn, and then a fourth approach that accepts the ambiguity in the regularization constants rather than aspiring to eliminate it. Whether any of the unique regularizations can be established as ‘correct’ (at least in some particular context) is of some physical interest, because it addresses the philosophical question of whether the ‘infinities’ in quantum field theory represent genuine physical ambiguities, or merely arise from a poor formulation of the technical mathematical problems that arise in the theory [11].

3. Principal value

In many cases integral (5) is improper but converges conditionally if the limit is taken in the most obvious, symmetrical way. Therefore, one defines

$$\langle \tilde{f}, \phi \rangle \equiv \lim_{\epsilon \downarrow 0} \int_{|\boldsymbol{x}-\boldsymbol{x}_0|>\epsilon} f(\boldsymbol{x})\phi(\boldsymbol{x})\, \mathrm{d}^n \boldsymbol{x}. \tag{10}$$

This form applies for an isolated singularity in any $\mathbb{R}^n$; in $\mathbb{R}^1$ it is more common to write

$$\langle \tilde{f}, \phi \rangle \equiv \lim_{\epsilon \downarrow 0} \left(\int_{-\infty}^{x_0-\epsilon} f(x)\phi(x)\, \mathrm{d}x + \int_{x_0+\epsilon}^{\infty} f(x)\phi(x)\, \mathrm{d}x \right). \tag{11}$$

This definition *usually does not work*, because the limit does not exist; but when it does, it gives a canonical extension $\tilde{f}$. On examples (7), the method obviously fails for $f(x) = \frac{1}{x^2}$ and $f(x) = \operatorname{cosech}^2 x$, where the left and right integrals do not cancel; but it works for $f(x) = \frac{1}{x}$ and

$$f(x) = \coth x \sim \frac{1}{x} + a_1 x + a_2 x^3 + \cdots.$$

The resulting regularized distribution $\tilde{f}$ is called the *principal value* of f and is 'officially' denoted as

$$\mathrm{PV}\left(\frac{1}{x}\right) \qquad \text{(for example)}.$$

This distribution is often written just as $\frac{1}{x}$ 'when there is no risk of confusion', but this practice is dangerous, since as we will see in the next section there are other reasonable regularizations of the function $\frac{1}{x}$ that are not equivalent to this one.

According to our basic principle, we must have for any such regularization

$$\langle \tilde{f}, \phi \rangle = \int_{-\infty}^{\infty} \frac{\phi(x)}{x}\,\mathrm{d}x \tag{12}$$

whenever this integral converges. Since ϕ is a smooth function, that happens if and only if $\phi(0) = 0$. If we now observe that a general function in $\mathcal{D}$ can be written as the sum of such a function that vanishes at the origin and a scalar multiple of some fixed function with $\phi(0) = 1$, it follows that the only freedom in the definition of $\tilde{f}$ is to add a multiple of the delta function:

$$\tilde{f} = \mathrm{PV}\left(\frac{1}{x}\right) + c\delta(x). \tag{13}$$

It is sometimes argued that since the original function $f(x) = \frac{1}{x}$ is odd, the regularization ought to be odd; this forces the choice $c = 0$.

4. Analytical continuation

Consider the example

$$f(x) = x_+^\lambda \equiv \begin{cases} x^\lambda & x > 0 \\ 0 & x > 0. \end{cases} \tag{14}$$

If $\mathrm{Re}\,\lambda > -1$, then f is in $L^1_{\mathrm{loc}}(\mathbb{R})$ and therefore defines a distribution unambiguously. For a fixed ϕ the quantity $\langle x_+^\lambda, \phi(x)\rangle$ can be analytically continued as a function of the complex variable λ, and one thus obtains a finite answer for any λ that is not a negative integer. (The same strategy works for r^λ when $r \equiv \|\boldsymbol{x}\|$ in $\mathbb{R}^n$, and for ρ_+^λ where

$$\rho^2 \equiv \|\boldsymbol{x}\|^2 - (ct)^2$$

as appears in solutions of the wave equation [4].)

One can define x_-^λ (x^λ restricted to negative x) in the same way. After some complex analysis that we do not go into in detail (see [5]), one finds that $|x|^\lambda$ is thereby defined for all λ except $-1, -3, -5, \ldots$ (because the poles of x_+^λ and x_-^λ cancel at the even negative integers). Similarly, $\mathrm{sgn}\, x|x|^\lambda$ is defined for all λ except $-2, -4, -6, \ldots$. Putting these two results together, we see that x^{-n} is now *canonically* defined for *all* integers n.

This definition extends immediately to any functions given by Laurent series, such as coth or cosech^2, since [6] as $x \to 0$

$$\coth x \sim \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \cdots \tag{15}$$

$$\mathrm{cosech}^2\, x \sim \frac{1}{x^2} - \frac{1}{3} + \frac{1}{15}x^2 + \cdots. \tag{16}$$

The leading, singular terms are defined as distributions by analytical continuation, and the remainders are nonproblematic.

Remark. *Differentiation* of a distribution is defined by

$$\langle f', \phi\rangle \equiv -\langle f, \phi'\rangle. \tag{17}$$

This identity is already true whenever f is a differentiable (and nonsingular) function, by virtue of integration by parts. (Recall that ϕ has compact support inside U, so there is no contribution from endpoints.) Definition (17) extends the identity to other distributions that are not 'classically' differentiable, such as step functions and similar discontinuous functions, whose derivatives contain delta functions. However, if f and f' are defined from singular functions by regularization, it is not immediately obvious that the distributional derivative (17) of f coincides with the regularization of the function f'; this must be checked. It is easy to check that indeed (when n is an integer)

$$(x^{-n})' = -nx^{-n-1} \quad \text{distributionally} \tag{18}$$

and therefore the desirable property (17) is preserved by the regularization by analytical continuation. In particular,

$$(\coth x)' = -\operatorname{cosech}^2 x \tag{19}$$

within this regularization. We shall have more to say on (19) in section 8.

Again, when the analytical regularization is understood as canonical, it is common to fail to mention 'this is a regularization', and to make no notational or verbal distinction between the function x^{-n} and the distribution x^{-n}. However, by comparing with the previous section one can see that this is dangerous. There is more than one way to reach x^{-n} by analytical continuation. Instead of continuing in the exponent, one could continue in a parameter added to the denominator:

$$\frac{1}{x} \equiv \lim_{\epsilon\to 0} \frac{1}{x + \mathrm{i}\epsilon}. \tag{20}$$

It is well known [5, (2.4.18)] that this limit *differs* from the principal value definition (and morever depends on the sign of ϵ):

$$\frac{1}{x + \mathrm{i}0} = \mathrm{PV}\left(\frac{1}{x}\right) - \mathrm{i}\pi\delta(x) \tag{21a}$$

$$\frac{1}{x - \mathrm{i}0} = \mathrm{PV}\left(\frac{1}{x}\right) + \mathrm{i}\pi\delta(x) \tag{21b}$$

(the limit $\epsilon \to \pm 0$ being understood). We have three natural but inequivalent definitions of $\frac{1}{x}$ as a distribution, demonstrating that at bottom regularizations are not unique.

5. Hadamard finite part

We have seen that the analytical continuation method is more general than the principal value definition but is somewhat arbitrary, since for each new function one must find a parameter to continue in. The next method is the most general and systematic of the canonical regularizations.

We recall (5) that ideally one would prefer

$$\langle \tilde{f}, \phi\rangle = \int f(x)\phi(x)\, \mathrm{d}x.$$

If f is singular, let us consider temporarily

$$\begin{aligned} F(\epsilon) &\equiv \int_{|x-x_0|>\epsilon} f(x)\phi(x)\,\mathrm{d}x \\ &= F_1(\epsilon) + F_0(\epsilon) \end{aligned} \tag{22}$$

where the division into the 'infinite part' F_1 and the 'finite part' F_0 is chosen so that

$$\lim_{\epsilon\downarrow 0} F_0(\epsilon) \quad \text{exists.} \tag{23}$$

Clearly this construction is not unique, since any finite piece of F_0 could be moved to F_1 without violating (23). However, consider a *fixed* family of functions $\Phi_1(\epsilon), \Phi_2(\epsilon), \ldots$ with

$$\lim_{\epsilon\downarrow 0} \Phi_j(\epsilon) = \infty$$

and with inequivalent singular behaviour as $\epsilon \downarrow 0$, and demand that

$$F_1(\epsilon) = a_1\Phi_1(\epsilon) + \cdots + a_K\Phi_K(\epsilon). \tag{24}$$

For example, if F has a Laurent expansion about $\epsilon = 0$, one would choose $\Phi_j(\epsilon) = \epsilon^{-j}$. *Relative to such a family*, the result for F_0 is unique, and can be taken to be the regularization $\tilde{f}$:

$$\langle \tilde{f}, \phi\rangle = \lim_{\epsilon\downarrow 0} F_0(\epsilon). \tag{25}$$

Of course, in any particular case one must choose the family large enough that (24) can be satisfied. There will be a standard choice containing negative powers $\epsilon^{-\alpha}$, functions $\epsilon^{-\alpha}\ln\epsilon$ and whatever other elementary functions are needed in an asymptotic expansion of F as $\epsilon \downarrow 0$. The resulting distribution $\tilde{f}$ is denoted by $\mathrm{Fp}(f(x))$.

Let

$$H(x) \equiv \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{26}$$

(the Heaviside function, often denoted by $\theta(x)$ in the physics literature). The Hadamard prescription applied to $f(x) = x^{-\alpha}H(x)$ defines a distribution

$$\mathrm{Fp}\left(\frac{H(x)}{x^\alpha}\right)$$

for *all* α; when $\alpha \neq 1, 2, \ldots$, this distribution coincides with $x_+^{-\alpha}$ as defined by analytical continuation in section 4. That is, for negative integer exponents, one *does* get a finite answer; but it is discontinuous in α. The Hadamard prescription amounts to discarding the pole terms in the analytical continuation, and whenever α decreases by one unit, another pole needs to be discarded.

This discontinuity complicates the formulae for derivatives. If k is a positive integer, one finds that [4, 5]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{Fp}\left(\frac{H(x)}{x^k}\right)\right) = -k\mathrm{Fp}\left(\frac{H(x)}{x^{k+1}}\right) + \frac{(-1)^k\delta^{(k)}(x)}{k!}. \tag{27}$$

Thus the derivative of the regularization is *not* the regularization of the derivative in this case! Similarly, one has the 'anomalous' scaling law [4, 5]

$$\mathrm{Fp}\left(\frac{H(\lambda x)}{(\lambda x)^k}\right) = \frac{1}{\lambda^k}\mathrm{Fp}\left(\frac{H(x)}{x^k}\right) + \frac{\ln\lambda(-1)^{k-1}\delta^{(k-1)}(x)}{\lambda^k(k-1)!}. \tag{28}$$

If k is replaced by α (not a positive integer), the strange last term in (28) is absent. Interestingly, this strange term in formula (28) is related to the appearance of logarithmic terms in spectral asymptotic developments [3, section 6].

The Hadamard method does not work well in several dimensions, unless the problem can be recast in one-dimensional form. For example, if $F(x, y)$ is singular at the origin, then

$$\mathrm{Fp}\int_0^a\int_0^b F(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

is problematic to define, because the order of evaluation of the iterated integral is significant. But singularities at isolated internal points or on smooth submanifolds are easier to handle [2, 12].

6. Oscillatory functions

Let f be a locally integrable function in $\mathbb{R}\backslash\{x_0\}$. In many cases the integral

$$\int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x \tag{29}$$

is divergent because the limit of

$$F(\epsilon) = \int_{-\infty}^{x_0-\epsilon} f(x)\phi(x)\,\mathrm{d}x + \int_{x_0+\epsilon}^{\infty} f(x)\phi(x)\,\mathrm{d}x \tag{30}$$

as $\epsilon \downarrow 0$ is oscillatory. Think, for instance, of $f(x) = |x|^{-\alpha}\sin x^{-1}$. In these cases it is still possible that the limit $\lim_{\epsilon\downarrow 0} F(\epsilon)$ exists in the *distributional* sense [10].

A function $g(x)$ defined for $x > a$ has the limit L in the distributional sense as $x \to a^+$, written as

$$\lim_{x\to a^+} g(x) = L \quad \text{distributionally} \tag{31}$$

if for each $\phi \in \mathcal{D}$ with $\operatorname{supp}\phi \subset (0, \infty)$ we have

$$\lim_{\eta\to 0}\langle g(\eta x + a), \phi(x)\rangle = L\int_{-\infty}^{\infty}\phi(x)\,\mathrm{d}x. \tag{32}$$

Alternatively, (31) holds if there exists a positive integer n and a primitive of order n of g, G, which satisfies $G^{(n)}(x) = g(x)$, $x > a$, and

$$\lim_{x\to a^+}\frac{n!G(x)}{(x-a)^n} = L. \tag{33}$$

Returning to (29), when we can define

$$\langle f, \phi\rangle = \lim_{\epsilon\downarrow 0}\left(\int_{-\infty}^{x_0-\epsilon} f(x)\phi(x)\,\mathrm{d}x + \int_{x_0+\epsilon}^{\infty} f(x)\phi(x)\,\mathrm{d}x\right) \quad \text{distributionally} \tag{34}$$

we say that the regularization was obtained by *distributional continuity.*

More generally, if we follow the Hadamard finite part method, as given by (22) and (24), and if the limit of the 'finite' part $F_0(\epsilon)$ does not exist in the ordinary sense but does in the distributional sense, we obtain the *distributional finite part technique.*

7. Taylor series surgery

So far we have pursued the goal of associating a unique, canonical distribution with every function, even though it is sometimes necessary to 'throw away infinities' to get there. An alternative philosophy is to accept the nonuniqueness in the distribution as an inherent feature of the problem, and to represent it by terms with arbitrary coefficients.

Let B be a neighbourhood of the singular point x_0. A manoeuvre similar in spirit to the Hadamard regularization (but sometimes technically easier) is to throw away 'bad' terms *before* evaluating the integrals. That is, we write

$$\langle \tilde{f}, \phi \rangle \equiv \int_B f(x) \left[\phi(x) - \sum_{j<k} \frac{\phi^{(j)}(x_0)}{j!}(x - x_0)^j \right] \mathrm{d}x + \int_{\mathbb{R}\setminus B} f(x)\phi(x)\,\mathrm{d}x \tag{35}$$

if

$$f(x) = O\big(|x - x_0|^\beta\big) \qquad \text{as } x \to x_0 \quad \text{with } \beta + k > 1. \tag{36}$$

(There is a similar definition in $\mathbb{R}^n$, with the condition $\beta + k > n$.) The introduction of B is necessary to avoid possible problems in the first term of (35) at infinity, since the powers $(x - x_0)^j$ are not legitimate (compact support) test functions.

The result depends on B, which is quite arbitrary. However, changing B changes $\tilde{f}$ only by terms proportional to the delta function and its derivatives. Thus such terms, up to $\delta^{(k-1)}$, should be regarded as present in $\tilde{f}$ with *undetermined*, *finite* coefficients. (In quantum field theory, these coefficients become renormalized coupling constants.)

8. The square of the hyperbolic cosecant

Recently [8, 9] Ford and O'Connell described a situation where physical considerations may lead one to define a regularization for a particular function that conflicts with the normal mathematical considerations guiding systematic regularization prescriptions. They argued in favour of the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}\coth x = -\mathrm{cosech}^2\, x + 2\delta(x). \tag{37}$$

This contradicts (19), but we have seen in connection with (27) that this kind of modification of a naive derivative relation is not necessarily wrong when singular functions are involved. One must examine carefully how the distributions on the two sides of the equation are defined.

The mathematical part of the argument of [8, 9] is that one should rewrite $\coth x$ (see (7)) as

$$\begin{aligned} \coth x &= \operatorname{sgn} x \left[1 + \frac{2}{\mathrm{e}^{2|x|} - 1} \right] \\ &= (2H(x) - 1) + F(x) \end{aligned} \tag{38}$$

$$F(x) \equiv \operatorname{sgn} x \frac{2}{\mathrm{e}^{2|x|} - 1}. \tag{39}$$

The derivative of $2H(x) - 1$ is $2\delta(x)$. The derivative of $F(x)$ is (at least for $x \neq 0$) equal to

$$F'(x) = \frac{-4}{(\mathrm{e}^x - \mathrm{e}^{-x})^2} \tag{40}$$

which is another way of writing $-\mathrm{cosech}^2\, x$, as indicated in (7).

Ford and O'Connell were motivated to derive (37) by the need to verify consistency between a quantum statistical mechanical quantity [7]

$$\frac{\mathrm{d}}{\mathrm{d}t}\coth[\pi k T(t - t')/\hbar] \tag{41}$$

and the corresponding classical statistical mechanical quantity

$$2\delta(t - t'). \tag{42}$$

Away from the singularity, (41) is pointwise equal to

$$-\frac{\pi kT}{\hbar}\operatorname{cosech}^2[\pi kT(t-t')/\hbar] \tag{43}$$

whose limit is equal to zero as $\hbar \to 0$. Therefore, the authors of [8] take the position that (42) arises solely from the extra term in (37). We would prefer to rewrite (37) as

$$\frac{\mathrm{d}}{\mathrm{d}x}\coth x = -(\operatorname{cosech}^2 x)_{\mathrm{qu}} + 2\delta(x) \tag{44}$$

where $(\operatorname{cosech}^2 x)_{\mathrm{qu}}$ is *defined* as the portion of $\frac{\mathrm{d}}{\mathrm{d}x}\coth x$ that represents quantum fluctuations about the classical mean value. The letter [8] is silent on how $\operatorname{cosech}^2 x$ is to be interpreted as a distribution, but the unpublished note [9] explicitly requires that

$$\int_{-\infty}^{\infty} \operatorname{cosech}^2 x\,\mathrm{d}x = 0 \tag{45}$$

implementing this physical requirement.

The quantum definition does not have some properties that one would naively expect a regularization prescription to have. In addition to violating (19), it is not linear, in the sense that the sum of two distributions is not necessarily the same as the regularization of the sum of the corresponding functions: As pointed out in [9], $-\operatorname{cosech}^2 x + x^{-2}$ is a smooth function, so its identification as a distribution is unambiguous (and contains no delta function), so when $(x^{-2})_{\mathrm{qu}}$ is defined in the usual way (cf section 4), it follows that $\int_{-\infty}^{\infty}(x^{-2})_{\mathrm{qu}}\,\mathrm{d}x = 0$ and hence

$$\left(-\operatorname{cosech}^2 x + \frac{1}{x^2}\right)_{\mathrm{qu}} = -(\operatorname{cosech}^2 x)_{\mathrm{qu}} + \left(\frac{1}{x^2}\right)_{\mathrm{qu}} + 2\delta(x). \tag{46}$$

Let us now consider how $\operatorname{cosech}^2 x$ would be defined as a distribution within the general mathematical frameworks discussed earlier in this paper. As we have seen, $\coth x$ has a natural definition as a principal value integral (and its derivative is then defined by (17)), but $\operatorname{cosech}^2 x$ is more problematic because formally it contains an *uncompensated* infinity. Let us examine closely formulae (38)–(40). It should be noted that $F(x)$ is no less singular at 0 than $\coth x$ is. (The most obvious qualitative difference between them is that $F(x) \to 0$ as $x \to \pm\infty$, whereas $\coth x$ approaches $\operatorname{sgn} x$ at infinity. It is not clear why this should introduce into the derivative a term concentrated at zero!) Therefore, the calculation of the distributional derivative of F as simply the functional derivative (40), without considering the possibility of a further delta term, is not convincing without further explanation. Now $\coth x$ and $\operatorname{cosech}^2 x$ possess the Laurent expansions (15) and (16) and therefore can be regularized in the standard way, by either analytical continuation or extraction of the Hadamard finite part, to distributions that satisfy(19):

$$\frac{\mathrm{d}}{\mathrm{d}x}\coth x = -(\operatorname{cosech}^2 x)_{\mathrm{st}}. \tag{47}$$

This standard definition of cosech^2 does not have the awkward properties noted above in connection with differentiation and addition.

In contrast, the behaviour of $F(x)$ at 0 is more complicated:

$$F(x) \sim \frac{1}{x}[1 - |x| + O(x^2)] = \frac{1}{x} - \operatorname{sgn} x + \cdots. \tag{48}$$

In view of the term $-\operatorname{sgn} x$ in (48), any 'natural' regularization of F contains such a term, and the resulting F' will contain a term $-2\delta(x)$.

In summary, it is indisputable that the derivatives of coth and F differ by a term 2δ; it is also indisputable (see below) that $(\coth x)'$ itself gives rise to such a term in a certain limit.

Whether the delta term appears explicitly in a formula for $(\coth x)'$, (47) or (44), depends on the somewhat arbitrary choice between the 'standard' and 'quantum' definitions of cosech^2 as a distribution.

As we have just seen, in the standard way of thinking the delta function is part of the derivative of F, not that of coth. Moreover (and consequently), when cosech^2 and F' are defined as distributions by any of the standard regularizations, the delta term belongs to (40), not to (37). *Thus (19) is correct ((37) is not) when the most natural and standard definitions are used.* (It is clear from section 7 that one is free to adopt the definition that makes (37) correct, but then one must use that definition in any calculations involving $\operatorname{cosech}^2 x$ as a distribution. Note that (46) suggests that $-(\operatorname{cosech}^2 x)_{\mathrm{qu}}$ consists of a term $-2\delta(x)$ plus something smoother.)

It remains to explain how delta function (42) arises out of the small-$\hbar$ limit of (43) when the latter is given its standard definition as a distribution. Let us take $t' = 0$ (without loss of generality) and set $\lambda = \pi kT/\hbar$. Then we are interested in the distributional limit

$$\lim_{\lambda\to\infty} -\lambda(\operatorname{cosech}^2(\lambda t))_{\mathrm{st}}. \tag{49}$$

Because $\operatorname{cosech}^2 x$ vanishes exponentially fast at infinity, it belongs to the class of distributions to which the *moment asymptotic expansion* [5, section 3.3] applies:

$$\lambda f(\lambda t) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(t)}{n!\lambda^n} \qquad \text{as } \lambda \to \infty \tag{50}$$

where

$$\mu_n \equiv \langle f(x), x^n \rangle. \tag{51}$$

Thus limit (49) is

$$-\mu_0 \delta(t) \tag{52}$$

where (recall (15))

$$\begin{aligned} \mu_0 &= \int_{-\infty}^{\infty} \left(\operatorname{cosech}^2 x - \frac{1}{x^2} \right) \mathrm{d}x \\ &= \lim_{M\to\infty} \left\{ \left[-\coth x + \frac{1}{x} \right]_{-M}^{0} + \left[-\coth x + \frac{1}{x} \right]_{0}^{M} \right\} \\ &= -2. \end{aligned} \tag{53}$$

Thus (42) is reproduced. *The correct classical expression is already contained in the semiclassical limit of* $-(\operatorname{cosech}^2 x)_{\mathrm{st}}$*; it is not necessary to add it on by hand.*

An alternative route from (41) to (42) starts from the asymptotic development of $\coth \lambda x$ as $\lambda \to \infty$. The moment asymptotic expansion does not apply to this function since $\coth x$ does not vanish at infinity; however, from (38) and (39) we can write

$$\begin{aligned} \coth \lambda x &= \operatorname{sgn} \lambda x + F(\lambda x) \\ &= \operatorname{sgn} x + F(\lambda x) \qquad \lambda > 0 \end{aligned} \tag{54}$$

and therefore

$$\coth \lambda x \sim \operatorname{sgn} x + \sum_{n=0}^{\infty} \frac{\tilde{\mu}_n \delta^{(n)}(x)}{n!\lambda^{n+1}} \qquad \text{as} \quad \lambda \to \infty \tag{55}$$

where

$$\tilde{\mu}_n = \langle F(x), x^n \rangle \tag{56}$$

are the moments of the distribution F. (The moment asymptotic expansion applies to F because it is of exponential decay at infinity.) At this point we observe that distributional asymptotic expansions can be differentiated [5] to obtain from (55) the formula

$$-\lambda \operatorname{cosech}^2 \lambda x \sim 2\delta(x) + \sum_{n=0}^{\infty} \frac{\tilde{\mu}_n \delta^{(n+1)}(x)}{n!\lambda^{n+1}} \qquad \text{as} \quad \lambda \to \infty \tag{57}$$

and consequently

$$\lim_{\lambda \to \infty} -\lambda \operatorname{cosech}^2 \lambda x = 2\delta(x). \tag{58}$$

(Incidentally, comparison of (50) and (57) shows that $\tilde{\mu}_n = \mu_{n+1}/(n+1)$, but this formula is not needed to make the point.)

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